

Stability of Calabi flow near an extremal metric

Hongnian Huang and Kai Zheng

Abstract

We prove that on a Kähler manifold admitting an extremal metric ω and for any Kähler potential φ_0 close to ω , the Calabi flow starting at φ_0 exists for all time and the modified Calabi flow starting at φ_0 will always be close to ω . Furthermore, when the initial data is invariant under the maximal compact subgroup of the identity component of the reduced automorphism group, the modified Calabi flow converges to an extremal metric near ω exponentially fast.

1 Introduction

Let M be a Kähler manifold and Ω be the Kähler class in $H^2(M, R) \cap H^{1,1}(M, C)$. By the $\partial\bar{\partial}$ -lemma, any Kähler metric ω_φ in Ω can be written as

$$\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$$

for some smooth real valued Kähler potential φ . The space of Kähler metrics is defined by

$$\mathcal{H} = \{\varphi \in C^\infty(M, R) | \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

Donaldson [18], Mabuchi [21] and Semmes [22] independently defined a Weil-Peterson typed metric on \mathcal{H} , under which it becomes a non-positively curved infinite dimensional symmetric space. Chen [5] proved that any two points in \mathcal{H} can be connected by a $C^{1,1}$ geodesic and that \mathcal{H} is a metric space, which verify two of the Donaldson's conjectures.

In order to tackle the existence of constant scalar curvature Kähler metric (cscK) problem, Calabi [2][3] introduced a well-known functional

$$Ca(\varphi) = \int_M S(\varphi)^2 \omega_\varphi^n,$$

where $S(\varphi)$ is the scalar curvature of ω_φ . The critical point of this Calabi functional is called an extremal Kähler metric. Calabi discovered that an extremal Kähler metric is a cscK if and only if the Calabi-Futaki invariant

is equal to zero. Later, he suggested that one may study the gradient flow of the K-energy to search for the cscK. This flow is defined as

$$\frac{\partial \varphi}{\partial t} = S - \underline{S} \quad (1)$$

and it decreases the Calabi energy. Since it is a fourth order equation, the maximal principle fails. In [19], Donaldson proposed a programme to study the convergence of the Calabi flow. On a Riemannian surface, P. Chruściel [17] proved that the flow exists for all time and converges to a cscK metric by using the Bondi mass. Later Chen [6] and Struwe [23] gave a different proof assuming the uniformization theorem. In Chen-Zhu [16], they removed the assumption of the uniformization theorem. For higher dimensions, the Calabi flow has been studied in Chen-He [8] [9] [10] and Tosatti-Weinkove [24]. In Chen-He [8], they proved that the Calabi flow can start from a $C^{3,\alpha}$ Kähler potential and become smooth immediately as $t > 0$. One defines the little Hölder space $c^{k,\alpha}$ to be the closure of smooth functions in the usual Hölder norm $C^{k,\alpha}$.

Theorem 1.1. *If $\omega_{\varphi_0} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_0$ satisfies $|\varphi_0|_{c^{3,\alpha}(M,g)} \leq K$, and $\lambda\omega < \omega_0 = \omega_{\varphi_0} < \Lambda\omega$, where K, λ, Λ are positive constants, then the Calabi flow initiating from φ_0 admits a unique solution*

$$\varphi(t) \in C([0, T], c^{3,\alpha}(M, g)) \cap C((0, T], c^{4,\alpha}(M, g))$$

for small $T = T(\lambda, \Lambda, K, \omega)$. More specifically, for any $t \in (0, T]$, there is a constant $C = C(\lambda, \Lambda, K, \omega)$ such that

$$t^{1/4}(|\dot{\varphi}(t)|_{c^{0,\alpha}}(M) + |\varphi(t)|_{c^{4,\alpha}(M)}) \leq C, |\varphi(t)|_{c^{3,\alpha}(M)} \leq C.$$

Remark 1.2. *He [20] shows that the Calabi flow can start from ω_φ where $\varphi \in c^{2,\alpha}(M)$.*

Theorem 1.3. *The solution obtained above belongs to $C^0([0, T], c^{3,\alpha}(M)) \cap C^0((0, T], C^\infty(M))$.*

Chen and He then use an energy argument to show that when Kähler manifolds admits a cscK ω and the initial Kähler potential is $C^{3,\alpha}$ small, the Calabi flow converges exponentially fast to a cscK nearby. In He [20], he improved this result for $C^{2,\alpha}$ small initial Kähler potentials.

In this short note, we prove a parallel theorem for extremal Kähler metrics using a different method from Chen-Ding-Zheng [7]. In that paper, they defined a flow called the pseudo-Calabi flow. They proved the short time existence from $c^{2,\alpha}$ initial Kähler potentials, the long time existence under uniform Ricci bounds and the stability near a cscK. Since the linearized operator of the pseudo-Calabi flow is not self-adjoint, they set up a unified frame to tackle the stability problem of the Kähler Ricci flow (c.f.

Chen-Zheng [15]), the Calabi flow and the pseudo-Calabi flow. This method strongly relied on the geometric structure of the space of Kähler metrics .

First, we will prove the long time existence of the Calabi flow.

Theorem 1.4. *On Kähler manifolds admitting an extremal metric ω and for any positive constant \mathcal{K}, λ , there is a small constant ϵ depending on $\omega, \mathcal{K}, \lambda$, such that for any Kähler potential φ_0 , if*

$$|\varphi_0|_{C^{2,\alpha}(M)} < \mathcal{K}, \quad \lambda\omega < \omega_{\varphi_0}, \quad \int_M |\varphi_0|^2 \omega^n < \epsilon,$$

then the Calabi flow exists for all time.

Next we want to study the modified Calabi flow. Let K be a maximal compact subgroup in the reduced automorphism group. Denote the corresponding Lie algebra of K by $h_0(M)$, which is the ideal of holomorphic vector fields with zeros. For any holomorphic vector field $\tilde{Y} \in h_0(M)$, denote $\tilde{Y} = Y - \sqrt{-1}JY$. Then there is a real function θ_Y such that

$$L_{\tilde{Y}}\omega = L_Y\omega = \sqrt{-1}\partial\bar{\partial}\theta_Y(t)$$

and

$$\int_M \theta_Y \omega^n = 0.$$

For an arbitrary metric

$$\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi,$$

the corresponding $\theta_{\tilde{Y}}(\varphi)$ is

$$\theta_{\tilde{Y}}(\varphi) = \theta_Y + \tilde{Y}(\varphi).$$

Following Futaki-Mabuchi [1], suppose $\tilde{X}, \tilde{Y} \in h_0(M)$, then the bilinear form

$$B(\tilde{X}, \tilde{Y}) = \int_M \theta_{\tilde{X}(\varphi)} \theta_{\tilde{Y}(\varphi)} \omega_\varphi^n$$

is independent of the choice of ω_φ in the Kähler class $\Omega = [\omega]$.

Let $\varphi(t)$ be a one parameter of Kähler potentials satisfying the Calabi flow equation and let $\sigma(t)$ be the holomorphic group generated by X , the real part of the extremal vector field \tilde{X} . Then $\sigma^*(t)\omega = \omega + i\partial\bar{\partial}\rho(t)$ satisfies the Calabi flow equation since

$$\frac{\partial \sigma^*(t)\omega}{\partial t} = L_X\omega(t) = i\partial\bar{\partial}(S(t) - \underline{S}).$$

Hence we can choose $\rho(t)$ to be a parameter of Kähler potentials satisfying the Calabi flow equation starting from 0.

Let $\psi(t) = \sigma(-t)^*(\varphi(t) - \rho(t))$. Notice that, by definition, $X = \sigma_*^{-1}(\frac{\partial}{\partial t}\sigma)$. So we obtain the modified Calabi flow,

$$\begin{aligned}\frac{\partial \psi}{\partial t} &= -X(\psi(t)) + \sigma(-t)^*\left(\frac{\partial \varphi(t)}{\partial t} - \frac{\partial \rho(t)}{\partial t}\right) \\ &= -X(\psi(t)) + \sigma(-t)^*(S(\varphi(t)) - \underline{S}) - (S(\omega) - \underline{S}) \\ &= S_\psi - \underline{S} - \theta_X - X(\psi) \\ &= S_\psi - \underline{S} - \theta_X(\psi)\end{aligned}$$

Theorem 1.5. *On Kähler manifolds admitting an extremal metric ω , for any \mathcal{K} -invariant Kähler potential φ_0 close to ω (in the sense of Theorem (1.4)), the modified Calabi flow exponentially converges to a nearby extremal metric.*

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2 Long time existence

First of all, we would like to give a rough estimate of the geodesic distance between any two Kähler potentials φ_0, φ_1 when

$$\omega_{\varphi_0}, \omega_{\varphi_1} < \Lambda\omega.$$

Lemma 2.1.

$$d(\varphi_0, \varphi_1) < C(\Lambda) \left(\int_M |\varphi_0 - \varphi_1|^2 \omega^n \right)^{\frac{1}{2}}$$

Proof. Let $\varphi_t = (1-t)\varphi_0 + t\varphi_1$ for $0 \leq t \leq 1$. Then

$$\begin{aligned}d(\varphi_0, \varphi_1) &\leq L(\gamma_t) \\ &= \int_0^1 \left(\int_M \left(\frac{\partial \gamma_t}{\partial t} \right)^2 \omega_{\gamma_t}^n \right)^{\frac{1}{2}} dt \\ &\leq \int_0^1 \left(\int_M (\varphi_0 - \varphi_1)^2 \omega_{\gamma_t}^n \right)^{\frac{1}{2}} dt \\ &\leq C(\Lambda) \left(\int_M |\varphi_0 - \varphi_1|^2 \omega^n \right)^{\frac{1}{2}}.\end{aligned}$$

□

We are ready to give a proof of Theorem 1.4.

Proof. Suppose that the conclusion fails, then there exist positive constants $\mathcal{K}, \lambda, \Lambda$ and a sequence of φ_s^0 such that

$$|\varphi_s^0|_{C^{2,\alpha}} < \mathcal{K}, \quad \lambda\omega < \omega_{\varphi_0^s} < \Lambda\omega, \quad \int_M |\varphi_s^0|^2 \omega^n < \frac{1}{s} \quad s = 1, 2, 3, \dots$$

By virtue of the short time existence theorem, we get a sequence of solutions $\varphi_s(t)$ satisfying the flow equation (1) with $\varphi_s(0) = \varphi_s^0$. Let T_s be the first time such that

$$|\varphi_s(T_s) - \rho(T_s)|_{C^{2,\alpha}(\omega(T_s))} = 2C \quad \text{holds}$$

or

$$\lambda\omega(T_s) < \omega_{\varphi_s(T_s)} < \Lambda\omega(T_s) \quad \text{fails,}$$

where the constant C is from Theorem 1.1. Then T_s is bounded from below for sufficiently large s . Otherwise there is a subsequence of $\varphi_s(T_s)$ converging to φ_∞ in the $C^{2,\alpha'}(\omega)$ sense, where $\alpha' < \alpha$. Notice that $\lambda\omega \leq \omega_{\varphi_\infty} \leq \Lambda\omega$, but that $\lambda\omega < \omega_{\varphi_\infty} < \Lambda\omega$ fails.

On the other hand, Lemma (2.1) shows that $d(0, \varphi_s^0) \rightarrow 0$ as $s \rightarrow \infty$. Since the distance function decreases under the Calabi flow, we have

$$d(\rho(T_s), \varphi_s(T_s)) \rightarrow 0$$

as $s \rightarrow \infty$. Let $\varphi_\infty(t)$ be one parameter potentials satisfying the Calabi flow equation initiating from φ_∞ . Then for $t_0 \geq t$,

$$\begin{aligned} d(\rho(t_0), \varphi_\infty(t_0)) &\leq d(\rho(t), \varphi_\infty(t)) \\ &\leq d(\rho(t), \rho(T_s)) + d(\rho(T_s), \varphi_s(T_s)) + d(\varphi_s(T_s), \varphi_\infty(t)). \end{aligned}$$

By Lemma (2.1), $d(\varphi_s(T_s), \varphi_\infty(t)) \rightarrow 0$ as $s \rightarrow \infty$ and $t \rightarrow 0$. Hence $\rho(t_0) = \varphi_\infty(t_0)$, which implies $0 = \varphi_\infty$, a contradiction.

Moreover, from Theorem 1.3, we obtain the higher order uniform bounds of the sequence of the solutions:

$$|\varphi_s(T_s) - \rho(T_s)|_{C^{k,\alpha}(\omega(T_s))} \leq C(k), \quad \forall k \geq 0.$$

Therefore we can choose a subsequence of $\phi_s = \sigma(-T_s)^*(\varphi_s(T_s) - \rho(T_s))$ so that

$$\phi_s \rightarrow \phi_\infty \text{ in } C^{k,\alpha}(\omega), \quad \forall k \geq 0,$$

and

$$|\phi_\infty|_{C^{2,\alpha}(\omega)} = 2C \quad (\text{or } \lambda\omega < \omega_{\phi_\infty} < \Lambda\omega \text{ fails}).$$

However, this contradicts the fact that $d(0, \phi_\infty) = 0$. \square

Corollary 2.2. *Given a Kähler potential φ_0 close to 0 in the sense of Theorem 1.4, then the modified Calabi flow stays in a neighborhood of 0. If φ_0 is \mathcal{K} -invariant, then the modified Calabi flow converges to an extremal metric nearby.*

Proof. That the modified Calabi flow stays in a neighborhood of 0 can be easily seen from the regularity Theorem 1.3. Notice that the Calabi flow decreases the Calabi energy, i.e.

$$\frac{\partial}{\partial t} Ca(\omega_\varphi) = -2 \int_M \mathcal{L}_\varphi(S_\varphi) S_\varphi \omega_\varphi^n,$$

where \mathcal{L}_φ is the Lichnerowicz operator with respect to ω_φ . It follows that we can take a sequence of $t_j \rightarrow \infty$ such that

$$\lim_{j \rightarrow \infty} \int_M \mathcal{L}_{\varphi(t_j)}(S_{\varphi(t_j)}) S_{\varphi(t_j)} \omega_{\varphi(t_j)}^n = 0.$$

Then there is a subsequence of t_j such that $\psi(t_j)$ converges to a potential ψ_∞ in C^∞ and

$$\int_M \mathcal{L}_{\psi_\infty}(S_{\psi_\infty}) S_{\psi_\infty} \omega_{\psi_\infty}^n = 0.$$

Hence ω_{ψ_∞} is an extremal metric. If φ_0 is \mathcal{K} -invariant, then ψ_∞ is a fixed point under the modified Calabi flow and the modified Calabi flow decreases the geodesic distance between $\psi(t)$ and ψ_∞ . Hence the flow converges to ψ_∞ . \square

3 Exponential decay

We define the modified Calabi energy as

$$\widetilde{Ca}(\psi) = \int_M (S(\psi) - \underline{S} - \theta_X(\psi))^2 \omega_\psi^n.$$

The evolution of the modified Calabi energy along the modified Calabi flow is

$$\begin{aligned} \partial_t \int_M \dot{\psi}^2 \omega_\psi^n &= \int_M (2\dot{\psi}\ddot{\psi} + \dot{\psi}^2 \Delta_\psi \dot{\psi}) \omega_\psi^n \\ &= 2 \int_M \dot{\psi} (\dot{S}_\psi - \dot{\theta}_X(t) - \dot{\psi}_i \dot{\psi}^i) \omega_\psi^n \\ &= 2 \int_M \dot{\psi} (-L\dot{\psi} + \dot{\psi}^i S_i - \dot{\theta}_X(t) - \dot{\psi}_i \dot{\psi}^i) \omega_\psi^n \\ &= 2 \int_M \dot{\psi} (-L\dot{\psi} + \dot{\psi}^i (\dot{\psi}_i + \theta_X(t)_i) - \dot{\theta}_X(t) - \dot{\psi}_i \dot{\psi}^i) \omega_\psi^n \\ &= 2 \int_M \dot{\psi} (-L\dot{\psi} + \dot{\psi}^i \dot{\psi}_i + X(\dot{\psi}) - X(\psi) - \dot{\psi}_i \dot{\psi}^i) \omega_\psi^n \\ &= -2 \int_M \dot{\psi} L\dot{\psi} \omega_\psi^n. \end{aligned}$$

In this computation we use the identities

$$\dot{\psi}^i(\theta_X i + (X(\psi))_i) = \dot{\psi}^i(\theta_X(t))_i = X(\dot{\psi}).$$

The modified Calabi-Futaki invariant

$$\tilde{F}(\tilde{Y}) = F(\tilde{Y}) - B(\tilde{X}, \tilde{Y}) = \int_M \theta_{\tilde{Y}}(\psi)(S - \underline{S} - \theta_{\tilde{X}}(\psi))\omega_\psi^n \quad (2)$$

is equal to zero when M admits an extremal metric ω . This shows that the direction of the modified Calabi flow is always perpendicular to the kernel of the Lichnerowicz operator. To obtain exponential convergence, one needs to give a uniform lower bound of the the first eigenvalue of L_t along the modified Calabi flow. More precisely, we have the following lemma which is similar to Chen-Li-Wang [11].

Lemma 3.1. *Along the modified Calabi flow, there is a positive constant $\lambda > 0$ such that for sufficiently large t and for any*

$$f \in A_t = \{f \in C_R^\infty(M) \mid \int_M f \omega_{\psi(t)}^n = 0 \text{ and } \int_M \theta_Y(t) f \omega_{\psi(t)}^n = 0, \forall \tilde{Y} \in h_0(M)\},$$

we have

$$\int_M L_t(f) f \omega_{\psi(t)}^n \geq \lambda \int_M f^2 \omega_{\psi(t)}^n.$$

Proof. If not, there must be a sequence $\psi_s = \psi(s)$ and f_s such that

$$\int_M |(f_s)_{ij}|^2 \omega_{\psi_s}^n < \frac{1}{s}; \int_M f_s^2 \omega_{\psi_s}^n = 1; \int_M f_s \omega_{\psi_s}^n = 0. \quad (3)$$

Since the C^l norm of ψ_s is uniformly bounded for any $l \geq 0$. Using the Ricci identity

$$\int_M |(f_s)_{i\bar{j}}|^2 \omega_{\psi_s}^n = \int_M |(f_s)_{ij}|^2 \omega_{\psi_s}^n + \int_M R^{i\bar{j}}(f_s)_i (f_s)_{\bar{j}} \omega_{\psi_s}^n$$

and the interpolation inequality, we conclude that f_s are uniformly $W^{2,2}$ bounded. So we can pass to the limit and get

$$\int_M |(f_\infty)_{ij}|^2 \omega_{\psi_\infty}^n = 0; \int_M f_\infty^2 \omega_{\psi_\infty}^n = 1; \int_M f_\infty \omega_{\psi_\infty}^n = 0. \quad (4)$$

Since in local coordinates, $\uparrow \bar{\partial} f_\infty$ is holomorphic in the weak sense, f_∞ is smooth indeed. From the assumption of A_t we have

$$\int_M \theta_Y(\psi_\infty) f_\infty \omega_{\psi_\infty}^n = 0, \forall \tilde{Y} \in h_0(M).$$

In particular, we may choose $\tilde{Y} = \uparrow \bar{\partial} f_\infty \in h_0(M)$. Hence,

$$\int_M f_\infty^2 \omega_{\psi_\infty}^n = 0.$$

This contradicts (4). \square

It is easy to see that $\widetilde{Ca}(\psi(t)) \leq Ce^{-\lambda t}$. To get exponential convergence of $\psi(t)$, we calculate the evolution formula for $\int_M |\nabla^k(\psi(t) - \psi_\infty)|^2 \omega^n$:

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_M |\nabla^k(\psi(t) - \psi_\infty)|^2 \omega^n \\
&= \int_M \nabla^k(S - \underline{S} - \theta_X(\psi)) * \nabla^k(\psi(t) - \psi_\infty) \omega^n \\
&= \int_M (S - \underline{S} - \theta_X(\psi)) * \nabla^{2k}(\psi(t) - \psi_\infty) \omega^n \\
&\leq \left(\int_M (S - \underline{S} - \theta(X))^2 \omega^n \right)^{1/2} \left(\int_M |\nabla^{2k}(\psi(t) - \psi_\infty)|^2 \omega^n \right)^{1/2} \\
&\leq C \|S - \underline{S} - \theta(X)\|_{L^2(\omega)} \\
&\leq C \|S - \underline{S} - \theta(X)\|_{L^2(\omega_t)} \\
&\leq Ce^{-\lambda_2 t}.
\end{aligned}$$

By the Sobolev embedding, we conclude that

$$\|\psi_t - \psi_\infty\|_{C^l(\omega)} \leq \|\psi_t - \psi_\infty\|_{W^{k,2}(\omega)} \leq Ce^{-\lambda_2 t}.$$

Hence we obtain the result stated in Theorem 1.5.

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Hongnian Huang
Centre interuniversitaire de recherches en geometri et topologie
Universite du Quebec a Montral
Case postale 8888, Succursale centre-ville
Montreal (Quebec), Canada
E-mail address: hnhuang@gmail.com

Kai Zheng
Academy of Mathematics and Systems Sciences
Chinese Academy of Sciences
Bei- jing, 100190, P.R. China.
E-mail address: kaizheng@amss.ac.cn